problem can be found by solving a system of six linear algebraic equations.

In the current approach, we apply the exact boundary conditions [Eqs. (14) and (18)] at $R=0$. Thus the segment [0,1] is defined as the computational domain. A uniform mesh of 20 linear elements is used.

We would also like to solve the problem with the standard finite element procedure, and see how well we can do. To this end, we approximate the semi-infinite cylinder by a long finite cylinder extending from $x=-d$ to $x=L$. With the total number of elements in the mesh unchanged (20 elements), there are three parameters to be considered: the additional length d, the approximate boundary condtions at $x=-d$, and the distribution of the 20 elements in $(-d,L)$. We assume that there are n equal elements in $(0,L)$ and $20-n$ equal elements in $(-d,0)$. Several boundary conditions can be used at $x=-d$, namely $u=\theta=0$ (clamped), $H=\theta=0$ (guided), $u=M=0$ (simply supported), or $H=M=0$ (free).

Selected solutions for the normal displacement are compared in Fig. 1. These include the exact solution, the one using the exact boundary condition, and three solutions for the long finite cylinder with $d=10$: a clamped and a simply supported cylinder with $n=5$ and a clamped cylinder with $n=15$. The displacement is plotted in the interval [0,0.4]. The finite element solution using the exact boundary condition is very accurate, whereas the three other approximate solutions are much less so. Additional experiments show that inaccurate results are obtained by the standard finite element method for other values of d and n and for other boundary conditions at $x=-d$ as well.

Conclusion

A numerical scheme has been suggested for the solution of problems involving the bending of long axisymmetrically loaded cylindrical shells. By using an exact relation between the unknown variables and their derivatives, which in some cases is identical to the so-called "edge stiffness" relation, it is possible to treat most of the domain analytically. Thus, only a small computational domain is left for discretization. The exact relation is combined as a boundary condition with the finite element method. Numerical tests show that this procedure is both more accurate and more efficient than the standard one. A similar procedure can be applied to problems involving long beams. The application of the method to general elastic shells will be considered in future work.

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Buckling of Single Layer Composite Cylinders Subjected to Lateral Pressure

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Nomenclature

E, E_x, E_y = Young's moduli
F_i = parameters in stability equation, $i = 1 - 6$
$G_{xy} = G$ = shear modulus
k = dimensionless shear modulus, $k = G(1 - \nu_1\nu_2)/E_x$
L = cylinder length
n = number of circumferential buckle halfwaves
q, q_c = lateral pressure, buckling lateral pressure
R = cylinder radius
t = cylinder thickness
U, V, W = axial, circumferential, and radial displacements
x, y, z = axial, circumferential, and radial coordinates
X = material orthotropy parameter, $X^2 = \nu_1/\nu_2 = E_x/E_y$
α, λ, ϕ = dimensionless parameters, $\alpha = t^2/12R^2$, $\lambda = \pi R/L$, $\phi = qR(1 - \nu_1\nu_2)/E_x t$
$\epsilon_x, \epsilon_y, \gamma_{xy}$;
$\sigma_x, \sigma_y, \sigma_{xy}$ = in-plane strains and stresses
ν, ν_1, ν_2 = Poisson's ratio
θ = circumferential angle

I. Introduction

ONE of the main design criteria for cylinders in their engineering applications is the stability (buckling-resistance) when subjected to external loads of the following type: a) axial compression, b) shear forces, c) external uniform lateral pressure.

The analytic solutions for cases a) and c), for buckling of a cylinder made from an isotropic material, have been presented by Timoshenko and Gere,[1] Chaps. 11.1 and 11.5, respectively. Subsequently, for the same cases, Jones[2] developed analytic solutions for composite materials.

The purpose of the present effort has been to solve the buckling problem for single layer, composite, circular cylinders having simply supported edges, when subjected to an external uniform lateral pressure, case c). The composite is treated as an orthotropic material whose principal material directions are coincident with the axial and hoop directions.

Two buckling types, bending and shear, with different types of behavior were found. The analytic results were verified by numerical calculations.

II. Equilibrium Equations

We use in our presentation here the equilibrium differential equations derived in Ref. 1 Sec. 11.5(d) with respect to forces and moments acting in a cylindrical shell subjected to lateral pressure. Using the definitions of the forces and the moments per unit length

$$N_i = \int_{-t/2}^{t/2} \sigma_i \, dz, \qquad M_i = \int_{-t/2}^{t/2} z\sigma_i \, dz \tag{1}$$

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for any direction i, Hooke's law for orthotropic materials

$$\sigma_x = E_x(\epsilon_x + \nu_2\epsilon_y)/(1-\nu_1\nu_2),\ \sigma_y = E_y(\epsilon_y + \nu_1\epsilon_x)/(1-\nu_1\nu_2)$$

$$\sigma_{xy} = G\gamma_{xy} \tag{2}$$

and strain expressions in the form of Eqs. (10-1) and (10-15) in Ref. 1, one obtains the following equilibrium differential equations

$$R^2\frac{\partial^2 U}{\partial x^2} + (k+\nu_2)R\frac{\partial^2 V}{\partial x\partial\theta} - R\nu_2\frac{\partial W}{\partial x} + \phi R\left(\frac{\partial^2 V}{\partial x\partial\theta} - \frac{\partial W}{\partial x}\right) + k\frac{\partial^2 U}{\partial\theta^2} = 0 \tag{3a}$$

$$R\nu_1(k+\nu_2)\frac{\partial^2 U}{\partial x\partial\theta} + \nu_1 kR^2\frac{\partial^2 V}{\partial x^2} + \nu_2\frac{\partial^2 V}{\partial\theta^2} - \nu_2\frac{\partial W}{\partial\theta} + \alpha\left[\nu_2\frac{\partial^2 V}{\partial\theta^2} + \nu_2\frac{\partial^3 W}{\partial\theta^3} + R^2\nu_1(2k+\nu_2)\frac{\partial^3 W}{\partial x^2\partial\theta} + 2R^2\nu_1 k\frac{\partial^2 V}{\partial x^2}\right] = 0 \tag{3b}$$

$$R\nu_1\nu_2\frac{\partial U}{\partial x} + \nu_2\frac{\partial V}{\partial\theta} - \nu_2 W - \alpha\left[\nu_2\frac{\partial^3 V}{\partial\theta^3} + R^2\nu_1(4k+\nu_2)\frac{\partial^3 V}{\partial x^2\partial\theta} + \nu_2\frac{\partial^4 W}{\partial\theta^4} + 2R^2\nu_1(2k+\nu_2)\frac{\partial^4 W}{\partial x^2\partial\theta^2} + \nu_1 R^4\frac{\partial^4 W}{\partial x^4}\right] = \nu_1\phi\left(W + \frac{\partial^2 W}{\partial\theta^2}\right) \tag{3c}$$

for the nonaxisymmetric part of the displacements caused by buckling. For isotropic materials the equations reduce to Eq. (11.5f) of Ref. 1.

III. Buckling Mode and Buckling Pressure Equations

Using the approach of Timoshenko and Gere,[1] we assume the following solution of Eqs. (3)

$$U = \bar{U}\sin(n\theta)\sin(\pi x/L), \qquad V = \bar{V}\cos(n\theta)\cos(\pi x/L)$$

$$W = \bar{W}\sin(n\theta)\cos(\pi x/L) \tag{4}$$

where $\bar{U}$, $\bar{V}$, $\bar{W}$ are constants, and $n = 1, 2, 3\ldots$ are integers, which define the shape of the buckling mode. At the simply supported edges ($x = \pm L/2$) the boundary conditions $N_x = V = W = M_x = 0$ are satisfied. Upon substituting Eq. (4) into Eq. (3), one obtains a system of linear equations with respect to $\bar{U}$, $\bar{V}$, and $\bar{W}$. The condition for buckling requires that the determinant of the equations equals zero, which leads to the equation

$$\phi(F_1 + F_2\alpha + F_3\phi) = F_4 + F_5\alpha + F_6\alpha^2 \tag{5}$$

where

$$F_1 = \nu_1(n^2-1)[\lambda^4\nu_1 k + \nu_2(1-2\nu_1 k - \nu_1\nu_2)n^2\lambda^2 + k\nu_2 n^4] + \lambda^4\nu_1\nu_2 k \tag{6a}$$

$$F_2 = \nu_1\{(n^2-1)(\lambda^2+kn^2)(\nu_2 n^2 + 2\nu_1 k\lambda^2) - n^4\lambda^2\nu_2(k+2\nu_2) - n^2\lambda^4\nu_1(4k^2 + 7k\nu_2 + 2\nu_2^2) + \nu_2^2 n^2\lambda^2 + 2\nu_1\nu_2 k\lambda^4 + \lambda^2 n^2(k+\nu_2)[\nu_2 n^4 + 2\nu_1(2k+\nu_2)\lambda^2 n^2 + \nu_1\lambda^4]\} \tag{6b}$$

$$F_3 = -\lambda^2 n^2\nu_1^2(k+\nu_2)(n^2-1) \tag{6c}$$

$$F_4 = \lambda^4 k\nu_1\nu_2(1-\nu_1\nu_2) \tag{6d}$$

$$F_5 = [\nu_2 n^4 + 2\nu_1(2k+\nu_2)\lambda^2 n^2 + \nu_1\lambda^4][\lambda^4\nu_1 k + \lambda^2 n^2\nu_2 \times (1 - 2k\nu_1 - \nu_1\nu_2) + k\nu_2 n^4] + \nu_2(\nu_2 n^2 + 2\nu_1 k\lambda^2) \times [\lambda^2(1-\nu_1\nu_2) + kn^2] - 2n^2\nu_2[\nu_2 n^2 + \nu_1 \times (3k+\nu_2)\lambda^2][\lambda^2(1-\nu_1 k - \nu_1\nu_2) + kn^2] \tag{6e}$$

$$F_6 = \nu_1\lambda^4(\lambda^2+kn^2)[2\nu_1 k\lambda^2 + n^2\nu_2(1-2\nu_1 k - \nu_1\nu_2)] \tag{6f}$$

For an isotropic material, Eq. (5) with parameters of Eq. (6) reduces to the form obtained in Ref. 1.

IV. Buckling Mode and Pressure Calculation

We will obtain now the value ϕ and the buckling pressure q_c as an approximate solution for Eq. (5). The buckling mode is defined by the value n that minimizes q_c. The following estimates will be used.

$$X^2 \sim 1 \text{ or } \nu_1 \sim \nu_2 \sim \nu \sim 1 \text{ and } E_x \sim E_y \sim E,$$

$$G \lesssim E \text{ or } k \lesssim 1 \qquad R \lesssim L \text{ or } \lambda \lesssim 1, \qquad t \ll R \text{ or } \alpha \ll 1 \tag{7}$$

Let us assume that

$$n^2 \gg 1 \tag{8}$$

This allows us to replace the large discrete variable n by a continuous variable. It can be shown that the terms ϕ^2, α^2, ϕ_α are able to be neglected in Eq. (5) which leads to the solution

$$\phi = (F_4 + F_5\alpha)/F_1 \tag{9}$$

Upon neglecting terms in Eqs. (6) that are small because of estimates of Eq. (7) and inequality of Eq. (8), one finds

$$F_1 = \nu_1\nu_2 n^4[(1-2\nu_1 k - \nu_1\nu_2)\lambda^2 + kn^2]$$

$$F_4 = \lambda^4 k\nu_1\nu_2(1-\nu_1\nu_2), \qquad F_5 = F_1 n^2/X^2 \tag{10}$$

Let us now consider two extreme cases:

$$kn^2 \gg \lambda^2 \qquad \text{and} \tag{11a}$$

$$kn^2 \ll \lambda^2 \tag{11b}$$

For simplicity we assume $\lambda \sim 1$ or $L \sim R$.

In case a),

$$F_1 = \nu_1\nu_2 kn^6, \qquad F_5 = \nu_2^2 kn^8 \tag{12}$$

Upon substituting the expressions of Eq. (12) and F_4 from Eq. (10), into Eq. (9) one obtains

$$\phi = \lambda^4(1-\nu_1\nu_2)/n^6 + \alpha\nu_2 n^2/\nu_1 \tag{13}$$

The following minimum value ϕ and corresponding continuous value n are obtained from Eq. (13)

$$n^2 = [3(1-\nu_1\nu_2)X^2/\alpha]^{1/4}\lambda; \qquad \phi = \frac{4}{3^{3/4}}(1-\nu_1\nu_2)^{1/4}\lambda(\alpha/X^2)^{3/4} \tag{14}$$

It is readily seen that our initial assumption of Eq. (8) is fulfilled since $\alpha \ll 1$ according to Eq. (7). Upon substituting Eq. (14) into Eq. (11a), we obtain

$$k^4 \gg \alpha \tag{15}$$

Upon substituting the relations for α, λ, ϕ given in the nomenclature into Eq. (14), we obtain the final form for the buckling mode and the buckling pressure

$$n^2 = \sqrt{6}\pi(1-\nu_1\nu_2)^{1/4}(\nu_1/\nu_2)^{1/4}\sqrt{R/t}\,R/L$$

$$q_c = \frac{2\pi}{3\sqrt{6}} E_y(1-\nu_1\nu_2)^{-3/4}(R/L)(t/R)^{5/2}(E_x/E_y)^{1/4} \qquad (16)$$

Moreover, upon combining Eq. (16) with Eqs. (8) and (11a), we obtain the inequalities

$$R^3 \gg L^2 t, \qquad G^2/E^2 \gg Rt/L^2 \qquad (17)$$

which apply to the case of the thin cylinder. This solution for q_c reduces in the case of an isotropic material to the equation recommended for engineering analysis by Roark and Young.[3]

For case b), which is specified by inequality of Eq. (11b), one obtains

$$n^2 = 2[3\pi^2 G(1-\nu_1\nu_2)R^4/E_y L^2 t^2]^{1/3}$$

$$q_c = \frac{(3\pi^2)^{1/3}}{4} E_y(G/E_y)^{1/3}\,[t^7/R^5L^2(1-\nu_1\nu_2)^2]^{1/3} \qquad (18)$$

Inequalities of Eqs. (8) and (11b) can be rewritten in the form

$$G/E \gg t^2/12R^2 \gg (G/E)^4 \qquad (19)$$

From Eq. (18) it can be seen that the buckling pressure q_c decreases when the shear modulus G decreases. Therefore this type of buckling for case b) can be called "shear" buckling. Similarly, from Eq. (16), it can be seen that q_c is dependent on the values of the Young moduli and the Poisson modulus but is not dependent on the shear modulus. Therefore the type of buckling described in case a) can be called "bending" buckling.

Finally, let us consider a third case c) for which the shear modulus is very small, namely

$$k \ll \alpha \qquad \text{or} \qquad G/E \ll t^2/12R^2 \qquad (20)$$

Substituting $k = 0$ into Eqs. (6) one obtains from Eq. (9)

$$\phi = \alpha F_5/F_1 = \frac{\alpha}{\nu_1}\,[\nu_2(n^2-1) + 2\nu_1\nu_2\lambda^2 + \nu_1\lambda^4/(n^2-1)] \qquad (21)$$

For $\nu_1 \sim \nu_2$ and $\lambda \lesssim 1$, as assumed in Eq. (7), and when $n = 2$, ϕ reaches its minimum value, which leads to

$$q_c = \frac{1}{4}\frac{E_y}{1-\nu_1\nu_2}\left(\frac{t}{R}\right)^3\left[1 + \frac{2}{3}\nu_1\left(\frac{\pi R}{L}\right)^2 + \frac{\nu_1}{9\nu_2}\left(\frac{\pi R}{L}\right)^4\right] \qquad (22)$$

Therefore, the buckling pressure is finite even for zero shear modulus.

V. Numerical Experiments and Concluding Remarks

A. Direct Numerical Solution

We will use Eq. (9) with relations of Eq. (6) as our starting point for obtaining the direct numerical solutions since only very small terms were neglected in its development from Eq. (5), and therefore it is applicable to all practical cases. In contrast, Eqs. (16), (18), and (22), which are a further simplification of Eq. (9) are accurate only for specific sets of values of the parameters, when the corresponding inequalities for cases a), b), and c) are fulfilled.

For our first example, we consider a case of bending buckling. The geometry and material parameters of the cylinder being considered are as follows:

$$R = 5, \qquad L = 30 \qquad (23)$$

$$t = 0.03 \qquad (24)$$

$$\nu_1 = 0.33\,X, \qquad \nu_2 = 0.33/X \qquad (25)$$

$$k = 0.335 \qquad (26)$$

The inequality of Eq. (11a), which corresponds to bending buckling is fulfilled.

As a second example of direct numerical solutions, we consider a cylinder with the same parameters as for the first example, except for t and k, which are now

$$t = 0.0949 \qquad k = 10^{-2} \qquad (27)$$

Here k is less than in Eq. (26) so as to fulfill the condition of Eq. (11b) for shear buckling. Comparison of these results with analytic values, obtained from Eqs. (16) and (18) shows a qualitative agreement for n and a quantitative agreement for ϕ.

B. Finite-Element Calculations

Finite-element (FE) calculations were performed using the ADINA computer program. Comparisons of the buckling mode and the buckling pressure obtained by FE calculations, with the values obtained by direct numerical calculations according to Eq. (9), are presented in Table 1 for bending and shear buckling types, respectively. The FE calculations were performed using the geometry and material parameters given in A above for the following sets of elastic moduli:

$$X = 1, \quad E_x = E_y = 2000 \quad \nu_{xy} = 0.33 \quad G = 752$$

(isotropic material)

$$X = 2, \; E_x = 4000, \; E_y = 1000, \; \nu_{xy} = 0.66, \; G = 1504$$

$$X = 0.5, \; E_x = 1000, \; E_y = 4000, \; \nu_{xy} = 0.165, \; G = 376 \qquad (28)$$

For the case of shear buckling, the comparison is given for the same sets of Young and Poisson moduli of Eq. (28). Shear moduli were calculated using the definition for k and Eq. (27) to be

$$G = 22.44, \qquad 44.9, \qquad 11.22 \qquad (29)$$

for cases a, b, c, respectively.

The reasonable agreement between the results obtained by direct numerical calculations, using Eq. (9), and by FE calculation supports the approach presented here.

Here the comparison of the buckling mode and the buckling pressure obtained analytically and by numerical methods is presented very briefly. Further details will be published separately.

Table 1 Buckling mode and pressure obtained by direct numerical calculation and by finite element method

		Direct numerical results		Finite element results	
Buckling type	X	n	$q_c \times 10^{-3}$	n	$q_c \times 10^{-3}$
Bending buckling	1	4	0.857	4	0.963
	2	5	0.640	5	0.657
	0.5	3	1.31	4	1.35
Shear buckling	1	3	11.7	3	13.4
	2	3	7.78	3	8.11
	0.5	2	12.1	2	15.0

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